OAC Invitational 2025

OAC Problem Writing Committee

10 October 2025

Invitational Round Instructions

1. The Invitational Round will be held over a duration of **3 days**:
   - **GMT (Greenwich Mean Time):** 12:00 noon, 10 Oct 2025 – 12:00 noon, 13 Oct 2025
   - **US Eastern Time (ET):** 08:00 a.m., 10 Oct 2025 – 08:00 a.m., 13 Oct 2025
   - **Indian Standard Time (IST):** 05:30 p.m., 10 Oct 2025 – 05:30 p.m., 13 Oct 2025

   The paper will be sent to all qualifiers by email at the start time. You must submit your solutions before the end of the above window.
2. The round will consist of a **subjective paper**. Each question requires a detailed written solution. The paper is of a total of **450** points.
3. Solutions must be submitted as either:
   - Scanned handwritten work (clear and legible), or
   - A PDF written using LaTeX.

   The use of **Word documents is strongly discouraged** due to formatting inconsistencies.
4. You may use the following tools if desired:
   - Desmos, GeoGebra, Google Sheets (or any equivalent spreadsheet software)
   - Mathematica
   - Custom code (in any programming language)

   For every question where such tools are used, you must:
   - Clearly mention all software used, and
   - Attach the corresponding code file, Mathematica notebook, or spreadsheet.
5. The use of the **internet to search for answers** or the use of **AI tools** for solving problems is strictly prohibited. Any violation will result in **immediate disqualification**.
6. This is an **individual contest**. Collaboration or answer-sharing is not allowed.
7. You must keep your **rough work and derivations** safely and **upload them along with your final answers**. The organizers may review these as part of the evaluation process.
8. Difficulty is subjective. It is possible that there are two problems worth similar points but one being more difficult that the other.
9. By starting the Invitational Round, you agree to abide by all the above rules in good faith.

All the best for the Invitational Round!

1 Am I precessing?? (Or just tripping?)

With just one day left until school exams, Ayush and Osish gave up on the idea of preparing for them and instead devised a plan that would allow them to pass the exam easily. Their plan: steal the papers and memorize all the answers beforehand. There was just one problem: the papers were stored encrypted on a $1\,\text{mm} \times 1\,\text{mm}$ memory card on the other side of the globe.

1.1 Planning the heist

Determined to pass the exams, they started building a satellite (which they named Runlin) that could read the contents of the hard disk, decrypt them, and send the questions back to them on Earth. Without any delay, they launched Runlin into a circular prograde orbit 600 km above the Earth's surface. However, lacking enough funding, they did not add any way to control the direction in which the satellite antenna points. Instead, they added a gyroscope inside the orbiter to ensure it keeps pointing in the same direction. Another trade-off they had to make was that the satellite could communicate only to points directly in front of the antenna, and could only steal the paper when it was at the zenith of the memory card. The orientation of Runlin's gyroscope during launch was such that it lay in the plane of its orbit.

For this problem, assume the Earth to be a uniform and ideal sphere of radius 6400 km and mass 6×10^{24} kg, isolated from all other bodies in space (gravitational effects of the Sun, Moon, and other planets do not affect Runlin). Take the time period for one rotation of the Earth around its axis to be 24 hours.

(a) For what fraction of its orbit can Runlin communicate with the Earthlings? **[2]**

(b) Ayush and Osish live in Nepal (28° N, 84° E), whereas their question papers are stored near the island of Fatu-Hiva (10° S, 134° W), a French overseas colony. The satellite was launched such that it went overhead both locations. How long at minimum after stealing the papers will Runlin first report back to Ayush and Osish? **[25]**

It turns out that parking a satellite in a circular orbit is a tough task. Instead of a circular orbit, Runlin now orbits in an elliptical orbit with semi-major axis $a = 7000$ km (same as before), but with an eccentricity of $e = 0.01$.

(c) What is the minimum time after stealing the papers that Runlin can report them back to Ayush and Osish? Assume that this is the orbit of Runlin for the rest of the problem. **[17]**

(d) Find the inclination of the orbit. **[3]**

Due to the long time delay, Ayush and Osish ended up failing the exam. The principal of the school expelled them, no university was willing to take them in, and no one wanted to hire them. Years passed, and they roamed the streets aimlessly, unemployed. Then, a thought struck Osish. He remembered Runlin, which was still in its orbit, unperturbed after all these years. He thought of using his invention to test Einstein and the General Theory of Relativity, and with this, prove to the world how capable they were.

1.2 Apsidal Precession

Apsidal precession refers to the precession of the apsis, or the nodes of the orbit. We take the equator to be the reference plane, and the vernal equinox to be the reference direction. At the time of launch, the longitude of the ascending node of the orbit was zero.

The Laplace-Runge-Lenz (LRL) vector of a particle moving under an inverse square central force given by the potential

$$V(r) = -\frac{k}{r}$$

is defined as

$$\vec{A} = \vec{p} \times \vec{L} - mk\hat{r}$$

where $\vec{p}$ is the linear momentum of the particle, $\vec{L}$ is the angular momentum of the particle, and m is the mass of the particle. A remarkable property of this vector is that it is a constant of motion and does not change with time.

(a) Prove that the LRL vector does not change with time for an inverse square force like Newtonian gravity. Find and interpret the magnitude and direction of the LRL vector. **[8]**

The potential due to a central mass in general relativity is given by the Schwarzschild potential:

$$V(r) = -\frac{GMm}{r} - \frac{GML^2}{mc^2r^3}$$

(b) Find the rate of precession of the LRL vector under the Schwarzschild potential (assume $r \gg L/mc$). With this, find the rate of precession of the apsis of the orbit. Hence, obtain the argument of periapsis and the longitude of the ascending node as functions of time t since launch. **[15]**

1.3 Geodetic Precession

Geodetic precession, also called De Sitter precession, is an effect caused by the curvature of spacetime on a vector carried along an orbiting body. This affects the orientation of Runlin's gyroscope, causing it to precess.

The Schwarzschild metric is given by

$$ds^2 = -\left(\frac{1}{1 - r_S/r}\right) dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + \left(1 - \frac{r_S}{r}\right) c^2 dt^2$$

where $r_S = 2GM/c^2$ is the Schwarzschild radius of the central mass. θ and ϕ are the polar coordinates. From now on, take the reference plane to be the plane of the orbit, such that $\theta = \pi/2$ for all points on Runlin's orbit. As Runlin goes around the Earth, its ϕ coordinate goes from 0 to 2π and repeats.

(a) Find the form of the metric in a frame rotating around the Earth (where the new azimuthal coordinate is ψ) with an angular velocity equal to the mean motion n of Runlin (the mean motion is defined as $n = \frac{2\pi}{T}$ where T is the orbital period). **[6]**

In relativity, instead of ordinary 3D vectors, we deal with 4D vectors, often referred to as 4-vectors. These are simply an extension of ordinary 3-vectors but with an additional time component (0-th component).

Consider the spin 4-vector of the gyroscope, representing the direction the gyroscope points to. Call it $X = (X^0, X^1, X^2, X^3)$. The superscript does not denote exponentiation but simply indexes the components (X^0 is the time component, X^1 is the radial component, X^2 is the θ component, and X^3 is the ϕ component).

Solving the geodesic equation

$$\frac{d^2x_\gamma}{ds^2} + \Gamma^\gamma_{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 0$$

one obtains (we spare you the math here) that the components of the spin 4-vector change as

$$
\begin{aligned}
dX^0 &= 0 \\
dX^1 &= -nag^{11}(1-e^2)X^3 du \\
dX^2 &= 0 \\
dX^3 &= nag^{33}(1-e^2)X^1 du
\end{aligned}
$$

where $g^{\alpha\beta}$ is the metric tensor, with the required components

$$
\begin{aligned}
g_{11} &= -\frac{1}{1-r_S/r} \\
g_{33} &= -r^2 - \frac{n^2a^4(1-e^2)^4 + 4n^2a^5(1-e^2)^5\left[1/r - 1/(a(1-e^2))\right]}{c^2 - 3GM/a}
\end{aligned}
$$

and u defined as $u = t - \frac{na^2(1-e^2)^2}{c^2-3GM/a}\psi$.

(b) The orbit of Runlin can be well approximated as Keplerian for a single revolution. Find the mean values of r and $1/r$ (averaged over the true anomaly θ) during one revolution. Thus, find the average values of g^{11} and g^{33}, and use these for the rest of the problem. **[6]**

In GR, one cannot simply compare components of the spin 4-vector at different points. The vector must be scaled appropriately by the metric tensor components to get the physical component of the spin 4-vector in a local Cartesian coordinate system attached to Runlin. The local Cartesian coordinate system is defined as follows:

- Origin at Runlin
- x-axis points radially outward from Earth
- y-axis is perpendicular to x, pointing in the direction of increasing ϕ

The components of the spin vector (x, y) in this frame are

$$
\begin{aligned}
x &= \sqrt{-g_{11}}X^1 \\
y &= \sqrt{-g^{33}}X^3
\end{aligned}
$$

(c) Find the equations for dx/dt and dy/dt. With this, find the angular velocity of the spin vector. Knowing the angular velocity in the local frame, find the precession rate of the spin vector. How much is this for Runlin? **[6]**

1.4 Lens-Thirring Precession

Ayush carefully measured the precession rate of Runlin's gyroscope but found it slightly different from what he had calculated, taking into account apsidal and geodetic precession. As it turns out, life is not that simple! The culprit, as Osish suggested, is another effect predicted by the General Theory of Relativity—Lens-Thirring precession.

Lens-Thirring precession is caused by frame dragging. If the central mass (Earth, in this case) is rotating, it drags spacetime with it, making spacetime twist and rotate. This effect causes the orientation of the gyroscope to precess, as seen by an observer far away.

Ayush and Osish were too tired to do all the calculations by hand, so they looked up the solution in a textbook:

$$
\vec{\Omega} = \frac{2G\vec{S}}{c^2r^3}
$$

where $\vec{\Omega}$ is the angular velocity vector of precession and $\vec{S}$ is the angular momentum vector of the rotating central body.

(a) Find the average rate of precession of Runlin's gyroscope due to the Lens-Thirring effect. **[3]**

The Lens-Thirring effect also causes apsidal precession. However, when compared to apsidal precession due to the Schwarzschild metric, it is negligible (because the Earth's angular velocity is very small) and can be ignored here.

1.5 Thomas Precession

Just as Ayush and Osish were about to take a breath of relief, they turned the page of their textbook and discovered another effect that can cause precession: Thomas precession.

Thomas precession is a purely kinematic effect arising from special relativity alone, leading to precession of a vector. The angular velocity of precession is

$$\vec{\omega} = \frac{1}{c^2}\frac{\gamma^2}{\gamma+1}\vec{a}\times\vec{v}$$

(a) Find the average rate of precession of Runlin's gyroscope due to Thomas precession. Compare this to that obtained from the Lens-Thirring effect. **[4]**

Reading further in the textbook, they discovered that Thomas precession is already accounted for in the calculation of the geodetic effect. Fortunately, this calculation was not that difficult!

1.6 Now what?

Ayush and Osish were ready to combine all their hard work and get the final estimate for the observed precession of Runlin's gyroscope. Since the precession rate is very small, they waited a thousand years for it to accumulate to a measurable value.

(a) Find the difference in the direction in which Runlin's gyroscope points after a thousand years since launch. **[10]**

Measuring Runlin's new direction, they found that it matched precisely what they had calculated from the General Theory of Relativity! They were overjoyed. Finally, all their centuries of hard work had borne fruit. They immediately published their findings, confirming Einstein's theory once again. Soon came fame, respect, and money. They got all they could ever want—except one thing. Ayush had always had a childhood dream of winning a Nobel Prize and of becoming Nepal's first laureate. He finally had the chance to fulfill his long-held dream. But the decision was left up to the Royal Swedish Academy of Sciences. What do you think? Will Ayush be able to bring a Nobel Prize home?

(b) Such an experiment—to measure the change in direction of a gyroscope due to geodetic and Lens-Thirring effects—has already been done. Do you know the name of this experiment? **[0]**

2 Hijacking the Royal Swedish Academy of Sciences

Riccardo, a budding scientist from Italy, also decided to submit his findings to the Royal Swedish Academy of Sciences, hoping to win the Nobel Prize. However, when he came to know that the jury—composed of Omar and Milly, two senior swedish physicists—are planning to choose Ayush and Osish for the prize, he became outrageous. Riccardo hired Gabriele and Francesco, two professional and reputed Italian mafias, to kidnap Omar and Milly.

That very night, Gabriele and Francesco carried out the abduction. When Omar and Milly regained consciousness, they found themselves chained to a pole, a gun pointed at their heads. Soon began a terrifying game of Russian Roulette. Desperate, Omar grabbed Milly's hand and activated the Royal Academy's top-secret invention—a randomized teleporter! The machine teleported the two to a random time in the past, and at a random location on Earth.

Omar and Milly find themselves completely lost in this new place. The buildings around them looked centuries old. Super lucky for them, they noticed a tall tower having both a clock and a sundial.

(a) Using the sundial, Milly measured the azimuth and altitude of the Sun to be $116°49'18.4''$ and $52°24'20.7''$ respectively. The clock read the zonal time of $14:00:00$. What are the possible locations Omar and Milly could be at? (Having precessed enough in the previous question, ignore effects of precession here) **[30]**

They soon met a kind man named Kresimir, who helped them navigate the area. Omar told him their story—how Riccardo was trying to corrupt the Nobel process by force. Determined to help, Kresimir called upon his close allies: Princess Nikolina and General Kento Kakutani, the head of the royal army.

Together, the five of them approached King Peter Andolšek, asking for his help. The king agreed and assembled an army of 15,000 soldiers to help them restore peace and justice. Their plan: to use the teleporter once more and return to September 2025, when Riccardo first seized control of the Royal Swedish Academy of Sciences.

However, in their haste, they forgot the most important rule of teleportation—they had to hold hands. As a result, only Omar, Milly, Kresimir, and Kakutani made it through. They found themselves back in the present day, but once again in a random location. They rushed to the nearest airport. However, they were shocked to find out that Gabriele and his gang had been waiting for them there. By now, Riccardo had completely taken over the Royal Swedish Academy of Sciences.

As the mafiosi advanced, Kakutani drew his katana and swiftly chopped them down. Seizing the moment, the group ran towards the terminal and boarded the first available flight.

Not knowing where they started from and where they're going, Milly started tracking the zenith distance of the Sun. She started measuring the positions 20 minutes after the takeoff, measured every 20 seconds for 100 minutes, and stopped 20 minutes before the landing. During this period, the plane was essentially moving along a great circle at a constant speed.

The data she collected can be found is *Milly.csv*. It contains two columns: time of measurement in seconds after after she started measuring, and zenith distance of the Sun.

(b) What is today's date according to Milly's measurements? **[20]**

(c) In which big (> 1M inhabitants) city did the flight start and in which big (> 1M inhabitants) city did it land? **[20]**

(d) Approximately at what local time did the plane take off and what is the time when the plane landed? **[20]**

Upon landing, Milly spots an old friend of hers — a former jury member named Cooline. The quartet explains the situation to her and together they form the BIG FIVE: Omar, Milly, Kresimir, Kento, and Cooline. Soon after, they take off for Sweden, determined to stop Riccardo once and for all.

3 Finding Riccardo

Without any delay, Cooline launches a satellite to track down Riccardo. Though, to determine the trajectory of the satellite, she realizes she can't simply apply Kepler's laws.

Kresimir, curious as to what the issue with using Kepler's laws is, approaches Cooline. She explains to him that when solving the Newton's $F = ma$ equation to obtain a solution for the trajectory of a satellite (Kepler's laws) in the gravitational field of the Earth we make the assumptions that:

- The only force acting on the body is the gravity of the Earth.
- The Earth is a perfect sphere with homogenous mass distribution.

However, in reality, these assumptions are not quite true. One must look at what happens when we allow more general forces and take into account the goofy shape of the Earth.

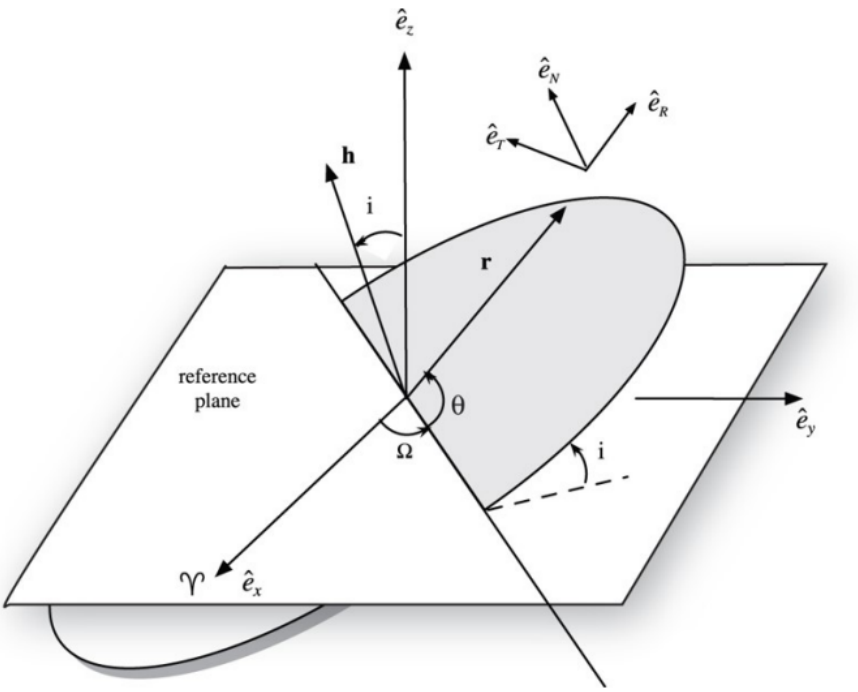
3.1 Generalized Perturbation Analysis

To solve problems effectively, we need a new set of basis vectors. The coordinates r (the Earth-satellite distance) and θ (angle between $\vec{r}$ and the line of nodes) will be used to describe the position of the satellite in its orbit. Suppose that the perturbative force per unit mass acting on the body is given by the vector

$$\vec{F} = R\hat{e}_R + N\hat{e}_N + T\hat{e}_T$$

Unit vectors $\hat{e}_R, \hat{e}_N$ and $\hat{e}_T$ are defined in the following way:

- $\hat{e}_R$ points along the $\vec{r}$ vector
- $\hat{e}_N$ points along $\vec{h}$ (specific angular momentum; $\vec{h} = \frac{\vec{L}}{m}$)
- The direction of the $\hat{e}_T$ vector is found by the product $\hat{e}_N \times \hat{e}_R$



To predict the trajectory of the satellite, Cooline has to express the quantities $\dot{a}, \dot{i}, \dot{\Omega}, \dot{\omega}$ and $\dot{e}$ (derivatives of the semi-major axis, inclination, right-ascension of the ascending node (RAAN for short), argument of perihelion and eccentricity of the orbit, respectively) in terms of the components R, N and T.

You may assume the orbit to be approximately keplerian over short time scales.

(a) Let ε be the specific orbital energy and μ the gravitational parameter GM. Express $\dot{a}$ in terms of ε, $\dot{\varepsilon}$, and μ **[2]**

(b) Express $\dot{e}$ in terms of e, h, $\dot{h}$, ε, and $\dot{\varepsilon}$. **[2]**

(c) Find $\dot{h}$ and $\dot{\varepsilon}$ in terms of coordinates r, θ, their derivatives, and the components of the force R, N, and T. **[6]**

(d) Express $\dot{e}$ and $\dot{a}$ in terms of a, μ, e, R, T, E_{ecc}, and f (hint: $\dot{\omega} \ll \dot{f}$). **[18]**

(e) Express $\dot{i}$ in terms of a, μ, e, N, and θ. You will need to use the transformation given below **[12]**

$$R_{RTN \to XYZ} = \begin{bmatrix} \cos\Omega\cos\theta - \sin\Omega\sin\theta\cos i & -\cos\Omega\sin\theta - \sin\Omega\cos\theta\cos i & \sin\Omega\sin i \\ \sin\Omega\cos\theta + \cos\Omega\sin\theta\cos i & -\sin\Omega\sin\theta + \cos\Omega\cos\theta\cos i & -\cos\Omega\sin i \\ \sin\theta\sin i & \cos\theta\sin i & \cos i \end{bmatrix}$$

Although a bit more complicated, one can also derive $\dot{\omega}$ and $\dot{\Omega}$. The results are given below:

$$\dot{\omega} = -\dot{\Omega}\cos i + \sqrt{\frac{a(1-e^2)}{e^2\mu}} \left(-R\cos f + T\frac{(2 + e\cos f)\sin f}{1 + e\cos f} \right)$$

$$\dot{\Omega} = \sqrt{\frac{a(1-e^2)}{\mu}} \frac{N\sin(\omega + f)}{\sin i(1 + e\cos f)}$$

3.2 The Goofy Shape of the Earth

Earth, sadly, is not a perfect sphere and looks more like the object below. This shape is termed as a geoid.

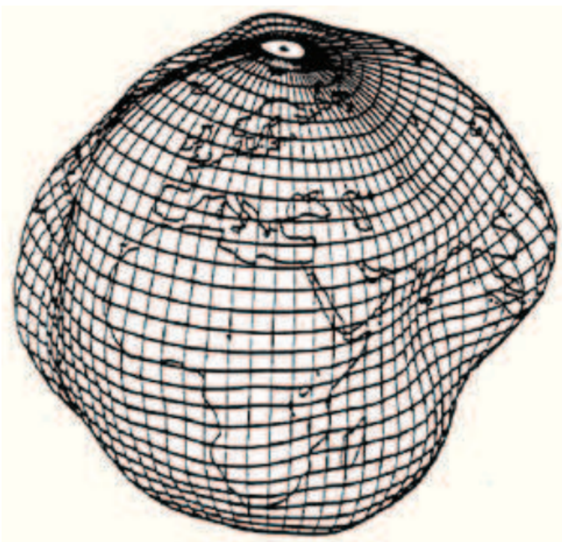
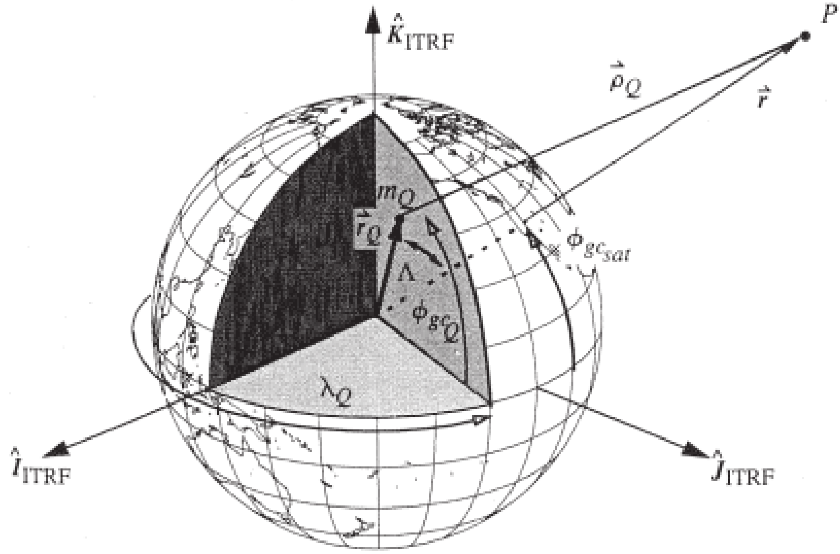


Figure 1: A Geoid (The unevenness is highly exaggerated)

This causes the radial gravity field to be not-so-radial anymore. To describe this, we first need the gravitational potential (per unit mass). This is given by

$$-U(\phi_{gc}, \lambda, r) = \frac{\mu}{r} + U_{zonal}(r, \phi_{gc}) + U_{sectorial}(r, \lambda) + U_{tesseral}(r, \phi_{gc}, \lambda)$$



The zonal contribution is given by

$$U_{zonal}(r, \phi_{gc}) = \frac{\mu}{r} \sum_{l=2}^{\infty} J_l \left(\frac{R_e}{r}\right)^l P_l(\sin \phi_{gc})$$

where R_e is the Earth's equatorial radius and $P_l(x)$ are the Legendre polynomials, given by Rodrigues' formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}((x^2 - 1)^l)$$

The coefficients J_l are obtained experimentally. Cooline decides to use the Geodesy data: $J_2 \approx 0.0010826$. She decides to analyze the effects only due to this $l = 2$ term, also called the J_2 effect.

(a) Express the components F_z and F_R of the perturbative force due to the J2 contribution. **[10]**

(b) Express the vector found in (a) in terms of RTN basis vectors. **[10]**

(c) Find $\langle \frac{d\Omega}{d\theta} \rangle$. The average is to be taken over the entire period of the orbit. By using this expression and by defining $\langle \frac{d\theta}{dt} \rangle = n$, express $\langle \dot{\Omega} \rangle$ in terms of n, J_2, a, e and i **[20]**

Although a bit tedious, we can also find $\langle \frac{d\omega}{d\theta} \rangle$. The result is given below

$$\langle \frac{d\omega}{d\theta} \rangle = -\langle \frac{d\Omega}{d\theta} \rangle \cos i + \frac{3 J_2 R_e^2}{2p^2} \left(1 - \frac{3}{2} \sin^2 i\right)$$

where p is the semi-latus rectum of the orbit.

(d) Using this equation, determine the inclination of the orbit for which $\langle \dot{\omega} \rangle$ is zero. **[5]**

Milly advised Cooline to set the inclination of the satellite to that found in (d) (She really did not want to deal with precession again). Setting the apogee of the satellite above Sweden, Cooline launched the satellite in a Tundra orbit, hoping to catch Riccardo soon.

4 Among the stars

The BIG FIVE found Riccardo — finally. Kento swiftly cut down all the guards with his katana. The team soon entered Riccardo's den, surrounding him from all sides. There, they discovered that he had imprisoned Ayush and Osish.

Just as they were about to capture him, Riccardo used his final weapon. He transformed himself into a star and, along with Ayush and Osish, disappeared into the cosmos. One stellar flare, and the Royal Academy would be gone forever.

To save the planet, the BIG FIVE had to act quick. There, Kresimir's knowledge and experience with stars came to the rescue. He quickly started analyzing the sky.

4.1 I see you

(a) There is a file named *Riccardo.jpg*. Find Riccardo, given he is in the field of view of the image. Circle or box the star with red color. **[5]**

4.2 Where are you??

Kresimir, not that familiar with the night sky, wasn't able to find Riccardo (Did you find him?). He resorted to another method: Analyze the light coming from the stars.

He measured the B magnitude, V magnitude and distance to several stars using an ideal CCD camera, with the filters being ideal; only allowing wavelengths of 440 nm by the B filter and 550 nm by the V filter to pass through. His recordings are given in three files. The data in these files is described as follows:

- *KresimirB.csv* contains the relative integrated flux measured by the CCD with the B filter.
- *KresimirV.csv* contains the relative integrated flux measured by the CCD with the V filter.
- *KresimirD.csv* contains the distance to the star (in parsecs) nearest to that cell.

For calibration, calibrate the data such that star near the bottom left, having distance 0.34922 pc, has color index $B - V = 2.07$ and bolometric correction of -5.48. Assume 100% quantum efficiency. The data has already been corrected for dark current and readout noise.

(a) Find a relation between the color temperature T and the color index $B-V$ of a star, assuming it to be an ideal blackbody radiator. (For this part assume the zero-point of both the magnitude systems is the same) **[8]**

(b) Identify and make a table of all the stars recorded, and calculate the color index of each star. Thus, find the color temperature of each star using the formula found in (a). **[15]**

(c) Now, calculate the temperature of the star using the following formula:

$$T = 4600\,\mathrm{K}\left(\frac{1}{0.92(B-V)+1.7} + \frac{1}{0.92(B-V)+0.62}\right)$$

Further, refine this temperature T to obtain T_{pred} using the following:

- If $B - V < 0$, $T_{\text{pred}} = T$
- If $0 < B - V < 0.3$, $T_{\text{pred}} = 3.3\,T - 24150\,\mathrm{K}$
- If $0.3 < B - V$, $T_{\text{pred}} = 13\,T - 186400\,\mathrm{K}$

Using this, calculate the luminosity of the stars:

$$\log L = 6.93 \log T_{\text{pred}} - 26.3$$

where L is in units of $L_\odot$ and T_{pred} is in Kelvin. **[15]**

(d) Find the absolute magnitude of each star, and their bolometric correction (BC) values. Plot this graph of BC vs. T. From this graph, which star could Riccardo be? **[15]**

4.3 How art thou, Riccardo?

Analyzing the details further, reveals the true position of Riccardo. Having located him, Kresimir started analyzing his activity. He did what a great physicist must do! Assume Riccardo to be perfectly spherical, made up of ideal polytropic hydrogen gas.

(a) Assuming hydrostatic equilibrium, find a differential equation relating the pressure $P(r)$ and density $\rho(r)$ of the gas, and the radial coordinate r. **[10]**

Let the polytropic equation of state of the gas be given by

$$P = K\rho^{1+\frac{1}{n}}$$

where K is a constant. Further, rewrite the density profile as $\rho(r) = \rho_c\theta(r)^n$, where $\theta(r)$ is zero at the surface, and unity at the center.

(b) What do these boundary conditions for $\theta(r)$ mean physically? What should $\theta'(0)$ be? **[3]**

(c) Substitute the expression for P and ρ into the equation you found in (a), and simplify. If you did everything correct, you will find that substituting $r = \alpha\xi$, where α is a constant, will eliminate any constants in the equation, and will yield an equation solely in θ and ξ. Find this. **[4]**

The equation between θ and ξ you found in (c) is not just a single equation, but rather a family of equations, one for each n.

(d) Find the solution $\theta(\xi)$, and thus $P(r)$ and $\rho(r)$ for $n = 0, 1$ and 5. Let ξ_R be the ξ corresponding to the surface of the star. What is the value of ξ_R for both these models? **[12]**

(e) Show that the radius R and total mass M of the star are related by

$$M \propto R^{\frac{n-3}{n-1}}$$

[7]

(f) Find the total energy of the star. **[8]**

Omar notices that Q31 of the OAC Open had a similar vibe as this question. He wondered if he will be able to solve it now.

4.4 Why are you oscillating?

Having modeled Riccardo as a polytropic star, Kresimir notices that Riccardo is oscillating! He immediately got to modeling the oscillations.

(a) Consider that the star expands symmetrically such that a particle at a radial coordinate of r is now at a radial coordinate of $r(1+\varepsilon)$ where $\varepsilon \ll 1$. Obtain the change in density $\delta\rho(r)$ in terms of the steady state density $\rho(r)$ and ε. **[5]**

(b) Assuming that for small fluctuations in density, the pressure P of a parcel of gas varies as $P = b\rho^\tau$ where τ is independent of radial coordinate. Obtain the change in pressure $\delta P(r)$ in terms of the steady state pressure $P(r)$, τ and ε. **[5]**

(c) Obtain an expression for the period of the oscillations. **[8]**

Doing further analysis on composition and structure of Riccardo, Kresimir concluded that its radius is $0.25R_\odot$, mass is $0.2M_\odot$ and effective temperature is 3200 K. Having found and fully understood the behavior of Riccardo, the BIG FIVE started planning what to do next.

5 Wait, where are Ayush and Osish?

Closely examining Riccardo, Kresimir found that Ayush and Osish had transformed into planets orbiting Riccardo. He wants to make sure the two are safe, and thus tries to determine if any of them are able to sustain life.

5.1 Determination of Orbits

Ayush turned into a rocky planet, having mass $M_1 = 1.20M_\oplus$ and radius $R_1 = 1.10R_\oplus$, orbiting Riccardo in a circular orbit.

(a) Ayush, to sustain life, should have an Earthlike zero-greenhouse equilibrium temperature of $255\,\mathrm{K}$ (assume Bond albedo $A_1 = 0.30$ for Ayush). Find the semi-major axis a_1 (in AU) at which he must orbit Riccardo to achieve $T_{\mathrm{eq}} = 255\,\mathrm{K}$. **[2]**

Osish too is a rocky planet, having an eccentric orbit with semi-major axis $a_2 = fa_1$ $(f > 1)$ and eccentricity e, lying on the same plane as Ayush's orbit. Assume the same zero-greenhouse radiative balance as in (a). Ignore albedo effects for Osish. Assume that the atmospheric pressure on Osish is sufficiently high wherever needed so that CO_2 remains liquid whenever the temperature lies within the liquid region of the phase diagram.

(b) Determine the pair (f, e) that yields the maximum possible orbital eccentricity while maintaining liquid CO_2 at every point along Osish's orbit. **[6]**

(c) Does Osish's orbit intersect Ayush's orbit? Answer Yes or No and justify briefly. **[3]**

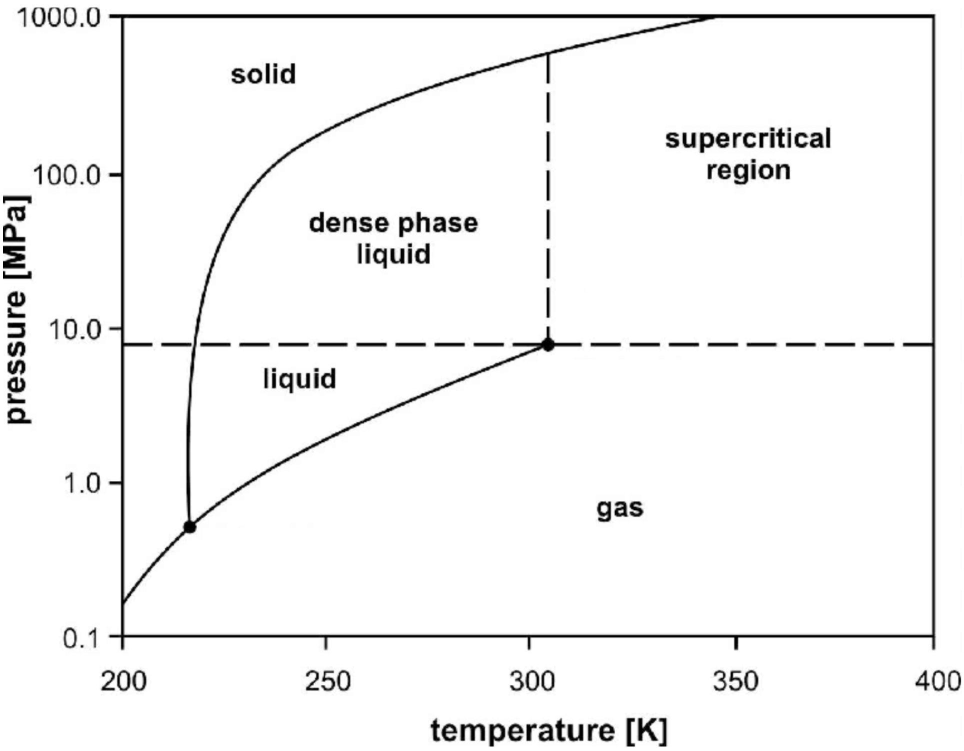


Figure 2: CO_2 pressure–temperature phase diagram.

5.2 Hydrogen Loss from an Ocean

Despite the unusual characteristics of Riccardo discussed in previous questions, we'll assume it to be a star of spectral class M5V. During early evolutionary stages of M5V stars, they emit high levels of XUV radiation. Assume the star remains in a saturated high-XUV state for the first 0.30 Gyr at a level

$$L_{\mathrm{XUV}} = 10^{-3} L_*$$

after which XUV-driven escape is negligible. Let the energy-limited escape efficiency be $\eta = 0.15$.

(a) Derive an energy-limited escape expression. Starting from a power balance between the useful fraction of the incident XUV power intercepted by a planetary cross-section and the power required to lift escaping mass out of the gravitational potential, derive an expression for the mass-loss rate $\dot{M}$. Your expression may include η, R_1, M_1, and the XUV flux at a_1. **[6]**

(b) Compute the XUV flux at Ayush's distance, then evaluate the corresponding $\dot{M}$ numerically for Ayush. **[2]**

(c) Using the hydrogen inventory implied by one Earth ocean and your $\dot{M}$, estimate the time τ_H required to remove all hydrogen (assume oxygen is retained). Compare τ_H to the 0.30 Gyr saturated epoch and state whether complete hydrogen loss of one ocean is expected during saturation. Briefly justify any approximations. (One Earth ocean mass is $M_{\mathrm{H_2O}} = 1.4 \times 10^{21}$ kg) **[7]**

5.3 Detectability of atmospheric O_2

Ayush, as observed from Earth, transits Riccardo. This enables transmission spectroscopy: during transit, Riccardolight skims through the Ayush's atmospheric limb. At wavelengths where the atmosphere is more opaque (e.g., the O_2 A-band near 0.76 µm), Ayush's effective radius is slightly larger than at nearby transparent wavelengths, making the in-band transit a bit deeper.

You are given that the mean molecular mass of the atmosphere is $29\,m_H$, limb (isothermal) temperature is 260 K and per-transit precision in a narrow O_2 band is 220 ppm.

(a) Derive the atmospheric scale height H from hydrostatic balance and the ideal gas law. Compute g from M_p and R_p, then evaluate H numerically for the given T and $\bar{\mu}$. **[3]**

(b) Using a thin-annulus geometric picture, derive an expression for the excess transit depth $\Delta\delta$ of a strong, saturated O_2 band in terms of R_1, $R_\star$, and H. Explicitly assume that a fixed factor of five scale heights contributes to the signal (i.e., take $N_H = 5$). In a strong absorption band, a thin, semi-opaque annulus of atmosphere around the planet blocks extra light. Evaluate $\Delta\delta$ (in ppm) using $N_H = 5$. **[8]**

(c) Assuming white noise with 20 ppm per transit in the O_2 band, derive how SNR scales with the number of identical transits N, and estimate how many are required for a 5σ detection of $\Delta\delta$. **[6]**

5.4 Globular Cluster

Omar suggests to look into Riccardo closely. Shocked to find, Riccardo was actually hiding in a globular cluster, outshining the rest of the members. Analysis of the cluster revealed something suspicious: All the stars of the cluster, when arranged by brightness in a decreasing order, and labeled $n = 1, 2, 3, \ldots$ ($n = 1$ being the brightest, and so on), their apparent magnitudes follow an arithmetic progression; Riccardo being the brightest.

(a) Suppose that the apparent magnitude of Riccardo as seen by Omar is m_0, and the total magnitude of the cluster is m_{tot}. Find out how the luminosities of the stars in the cluster are related to each other, in terms of m_0 and m_{tot}. **[2]**

(b) Omar decided to have some fun and goes a bit off track. He removes all the stars whose index satisfies $n \equiv 2 \pmod 5$ from his calculation, keeping all the other stars intact. Compute the combined apparent magnitude of these stars. **[2]**

(c) A patchy "zebra stripe" dust pattern dims odd-indexed stars by an additional 0.7 mag of extinction, while even-indexed stars suffer no extra extinction. Simultaneously, Omar removes all stars with $n \equiv 3 \pmod 5$. For the cluster under these two effects together, compute the combined apparent magnitude of the stars that remain. **[3]**

Kento stop Omar and reminds him about their mission. They must destroy Riccardo, rescue Ayush and Osish, and restore peace and order in this world.

6 The Journey Continues

Having ran out of ideas, and questions to go along with the lore, we ask you.

(a) Complete the story. How does the BIG FIVE manage to stop Riccardo, and free the world from his terror? **[?]**

This question carries no marks, but instead, there's a special prize for it.